Computation of Volume by triple integrals

$$\int_V f(x,y,z)\,dv = \iiint_R f(x,y,z)\,dx\,dy\,dz = \int_a^b \left[\int_{g(x)}^{h(x)} \left\{ \int_{\psi(x,y)}^{\phi(x,y)} f(x,y,z)\,dz \right\} dy \right] dx$$

As a particular case, $f(x,y,z)=1$,

$$\int_V dv = \iiint_R dx\,dy\,dz = \int_a^b \int_{g(x)}^{h(x)} \int_{\psi(x,y)}^{\phi(x,y)} dz\,dy\,dx \quad \text{—— (1)}$$

The integral $\int_V dv$ represents the volume V of R. The eq^n (1) may be used to compute V.

If (x,y,z) are changed to (u,v,w) we obtain,

$$\int_V dv = \iiint_R dx\,dy\,dz = \iiint_{R^*} J\,du\,dv\,dw \quad \text{—— (2)}$$

Taking, $(u,v,w) = (R,\phi,z)$ in (2)

we obtain, $\int_V dv = \iiint_R R\,dR\,d\phi\,dz$ —— (3) an expression for volume in terms of Cylindrical polar Coordinates.

Similarly, $\int_V dv = \iiint_R r^2 \sin\theta\,dr\,d\theta\,d\phi$ an expression for volume in terms of spherical polar Coordinates.

Note:

1. $R\,dR\,d\phi\,dz$ is the volume element in Cylindrical polar Coordinates and $r^2 \sin\theta\,dr\,d\theta\,d\phi$ is the Volume element in spherical polar Coordinates.

1. Find the Volume Common to Cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$

Sol^n In the given region 'z' Varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$ and 'y' Varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$. For $z=0$, $y=0$, x Varies from $-a$ to a.

$\therefore$ Required Volume $= V = \displaystyle\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \Big[z\Big]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\,dx$$

$$= 2\int_{-a}^{a}\left\{\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\right\}dx = 2\int_{-a}^{a}\Big[\sqrt{a^2-x^2}\,y\Big]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\Big[\sqrt{a^2-x^2}\;2\sqrt{a^2-x^2}\Big)\,dx = 4\int_{-a}^{a}(a^2-x^2)\,dx = 4\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a}$$

$$= 4\left[\left[a^3-\frac{a^3}{3}\right]-\left[-a^3+\frac{a^3}{3}\right]\right] = 4\left[2a^3-\frac{2a^3}{3}\right] = \frac{16a^3}{4}$$

2. Find the Volume of the Sphere $x^2+y^2+z^2=a^2$, Using triple integrals.

Sol^n Let 'V' be the volume of the sphere and $V=\iiint dx\,dy\,dz$ where 'R' is the spherical region of integration R. In this region R, 'r' varies from 0 to a, 'θ' varies from 0 to π

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and ϕ varies from 0 to 2π, where (r, θ, ϕ) are the spherical polar coordinates.

$$V = \iiint_R r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^a \int_0^\pi \left[(r^2 \sin\theta) \phi \right]_0^{2\pi} d\theta \, dr = \int_0^a \left[\int_0^\pi 2\pi r^2 \sin\theta \, d\theta \right] dr$$

$$= \int_0^a \left[-2\pi r^2 \cos\theta \right]_0^\pi dr = -2\pi \int_0^a r^2 (\cos\pi - \cos 0) dr = 4\pi \int_0^a r^2 dr$$

$$= 4\pi \left(\frac{r^3}{3} \right)_0^a = \frac{4\pi a^3}{3}$$

3. Find the volume bounded by the cylinder $x^2+y^2=4$ & the planes $y+z=3$ and $z=0$

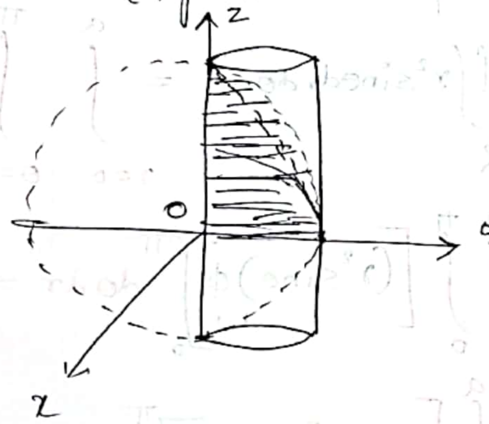
Sol

Here 'z' varies from 0 to $3-y$, 'y' varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$ and x varies from -2 to 2

$$V = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3-y} dz \, dy \, dx$$

$$= 12\pi$$

(4) Find the Volume Common to the sphere $x^2+y^2+z^2=a^2$ & the Cylinder $x^2+y^2=ax$



Sol^n

$x = R\cos\phi$, $y = R\sin\phi$,

$z = z$.

In terms of these coordinates, the eq^n of the sphere & cylinder becomes

$R^2+z^2=a^2$ and $R = a\cos\phi$.

Here, z varies from $-\sqrt{a^2-R^2}$ to $\sqrt{a^2-R^2}$, R varies from 0 to $a\cos\phi$ and ϕ varies from 0 to π

$$V = \int_{\phi=0}^{\pi} \int_{R=0}^{a\cos\phi} \int_{z=-\sqrt{a^2-R^2}}^{\sqrt{a^2-R^2}} R\,dz\,dR\,d\phi$$

$$= \frac{2a^3}{9}(3\pi-4)$$